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Time Evolution of Quasi-Stationary States*

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Abstract

In this paper we study the time evolution of prepared states in some quantum mechanical models, and discuss the probability of decay and the rate of energy dissipation and their dependence on the form of the interaction. First we consider solvable models with divergent matrix elements for the operator H^2, where H is the Hamiltonian of the system. We study two specific examples, one with well-defined eigenvalues and the other with renormalizable interaction. The time development of the initial state in the latter case depends on the cut-off parameter. In the second part of the paper, we show the possibility of existence of decaying states with long lifetime, where the amplitude of the initial state decreases like a Bessel function. In the third part, we determine the time development of a prepared state in a simple many-boson problem. Finally we study the problem of penetration of a wave packet through two phase-equivalent potential barriers, and we conclude that from the scattering phase shifts alone, it is not possible to determine the lifetime or the mode of decay of an unstable particle uniquely.

1. *Introduction*

If the initial state of a system is not an eigenstate of the Hamiltonian, it decays after a sufficient period of time into a mixture of the eigenstates of the system. The process of preparing the initial quasi-stationary state and its subsequent decay can be regarded as a scattering problem, and thus the Hamiltonian for the whole system, which describes the system before and after the formation of the unstable state, determines the way that the prepared state decays. However, if this prepared state has a sufficiently long lifetime that experimental observation may be made on it before its decay, the mode of formation becomes irrelevant, and the decay can be studied separately. While the relation of the decay of a quasi-stationary state to the scattering reaction by which the unstable state is prepared is important in some problems, we will, for simplicity, consider only those cases where the initial state has a sufficiently long lifetime.

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The other simplification that we want to make is to exclude the measuring apparatus from the total system. The extent of validity of this approximation is not clearly known at present. (For a detailed discussion of this subject the reader is referred to the papers of Ekstein and Siegert, 1971 and Fonda et al. 1973). With these assumptions we can state the problem in the following way: The initial conditions in an experiment described quantum mechanically are represented as the eigenstates of the unperturbed Hamiltonian. As a typical case, we assume that the system is prepared in a state $\phi_0(\mathbf{r})$ at $t = 0$, where $\phi_0(\mathbf{r})$ is an eigenfunction of the Hamiltonian without coupling. We may then ask what is the probability of finding the system in a definite state of the unperturbed Hamiltonian at a later time. But first we should investigate whether it is possible to prepare such a state as considered, since the initial state is not an eigenfunction of the total Hamiltonian. To have a realizable initial state, both the expectation value of the Hamiltonian and the magnitude of the rate of change of the initial state wave function must be finite. If the latter quantity is infinite the initial state $\phi_0(\mathbf{r})$ cannot be prepared, since it will change discontinuously into some other state or set of states. But one is led, in some cases, to consider such a state as the initial state for the system. For instance, in the Wigner-Weisskopf model (Wigner and Weisskopf, 1930), and in Martin's relativistic Lee model (Martin, 1963), the absorption and reemission of particles take place at the origin, thus the interaction is a δ-function in the coordinate space. Hence, in both models the matrix elements of H^2 with the unperturbed wave functions are divergent quantities.

We start in Section 2 by transforming the time-dependent Schrödinger equation to a set of differential-difference equations for the coefficients of expansion $(C_n(t))$ of the exact wave function in terms of the eigenvalues of the unperturbed system. These coefficients are directly related to the decay probability and the mean energy of the initial state. In Section 3 we discuss two models with δ-function interactions. For the first model we find a well defined eigenvalue equation (which, in fact, is identical to the eigenvalue equation in the Wigner-Weisskopf model), however, in the second model a cut-off is necessary to make the results finite. This renormalizable model is similar to the model Hamiltonian studied by Peres (Peres, 1969). In Section 4 we discuss the problem of the motion of a particle in a periodic potential and show that, under certain conditions, a prepared state in this system decays as a zeroth order Bessel function. Thus, the time evolution of this system is similar to that of the classical loaded chain of infinite length (Schrödinger 1914). In Section 5 we first consider a coupled channel problem, which is essentially Martin's model. Then we study the time dependent solutions of the Bassichis-Foldy many boson system (Bassichis and Foldy, 1964). We show that when the number of particles in the system, N, becomes very large and the strength of the two-particle force, g, tends to zero so that gN remains finite, the differential-difference equations of motion can be simplified, and the resulting set of equations may be solved exactly with the help of a fictitious Hamiltonian. Finally, we observe that phase equivalent potentials do not have identical penetration factors for a given wave packet (Section 6).

2. *Differential-Difference Equation for Expansion Coefficients*

The time-dependent Schrödinger equation

$$i\frac{\partial\psi(\boldsymbol{r},t)}{\partial t}=H\psi(\boldsymbol{r},t) \tag{2.1}$$

with the initial condition

$$\psi(\boldsymbol{r},t=0)=\phi_0(\boldsymbol{r}) \tag{2.2}$$

can be solved by taking the Fourier transform of every term in the equation (2.1) with respect to the variable t. The resulting eigenvalue equation

$$H\psi(\boldsymbol{r},E)=E\psi(\boldsymbol{r},E) \tag{2.3}$$

represents the stationary state solutions of the wave equation. If the characteristic functions of equation (2.3) are known and they are normalized according to the relation

$$\int\psi(\boldsymbol{r},E)\psi^*(\boldsymbol{r},E')d^3r=\delta(E\text{-}E') \tag{2.4}$$

then the solution of equation (2.1) at time t can be expressed as a superposition of the stationary states

$$\psi(\boldsymbol{r},t)=\int C_0(E)\psi(\boldsymbol{r},E)e^{-iEt}dE \tag{2.5}$$

In this equation $C_0(E)$ is the coefficient of expansion of $\phi_0(\boldsymbol{r})$ in terms of $\psi(\boldsymbol{r},E)$

$$\phi_0(\boldsymbol{r})=\int C_0(E)\psi(\boldsymbol{r},E)\,dE \tag{2.6}$$

The integrals in equation (2.5) and (2.6) extend over all possible eigenvalues of equation (2.3), if the spectrum of H is continuous. For systems having discrete eigenvalues, $\delta_{EE'}$ replaces the δ-function in equation (2.4) and summations over the eigenvalues take the place of integrals in equations (2.5) and (2.6). If we multiply equation (2.6) by $\psi^*(\boldsymbol{r},E)$ and integrate over the volume, then using equation (2.4) we find

$$C(E)=\int\phi_0(\boldsymbol{r})\psi^*(\boldsymbol{r},E)\,d^3r \tag{2.7}$$

Thus, the time-dependent wave function $\psi(\boldsymbol{r},t)$ can be written as

$$\psi(\boldsymbol{r},t)=\int d^3r'\,dE\phi_0(\boldsymbol{r}')e^{-iEt}\psi(\boldsymbol{r},E)\psi^*(\boldsymbol{r}',E) \tag{2.8}$$

In the case where $\phi_0(r)$ is a member of a complete set of functions $\phi_m(r)$ satisfying the orthogonality condition

$$\int\phi_m(r)\phi_n^*(r)\,d^3r=\delta_{mn} \tag{2.9}$$

we can expand $\psi(r,t)$ and $\psi(r,E)$ in terms of the set $\phi_m(r)$ and determine the coefficients of expansion. Thus

$$\psi(r,t)=\sum_m C_m(t)\phi_m(r) \tag{2.10}$$

and

$$\psi(r, E) = \sum_m C_m(E)\phi_m(r) \tag{2.11}$$

By substituting equation (2.10) and (2.11) in equations (2.1), (2.8) and (2.4), we find the following relations for C_n:

(a) The equation of motion can be written as

$$i\frac{dC_n(t)}{dt} = \sum_m H_{nm}C_m(t) \tag{2.12}$$

where

$$H_{nm} = \int \phi_n^*(r)H\phi_m(r)\, d^3r \tag{2.13}$$

(b) The relation between $C_n(t)$ and $C_n(E)$ is given by

$$C_n(t) = \int e^{-iEt}C_0^*(E)C_n(E)\, dE \tag{2.14}$$

(c) The normalization condition implies that

$$\sum_m C_m(E)C_m^*(E') = \delta(E-E') \tag{2.15}$$

and finally,

(d) The initial conditions follow from equation (2.10)

$$C_0(t=0) = 1, \qquad C_m(t=0) = 0 \qquad m \neq 0 \tag{2.16}$$

The probability that the state $\phi_0(r)$ has not decayed at the time t is given by $|C_o(t)|^2$. In addition to $|C_o(t)|^2$ there are two important quantities associated with the time evolution of a quasi-stationary system. First is the initial decay rate, Γ, defined by the equation

$$\Gamma = \left(\frac{d}{dt}|C_o(t)|^2\right)_{t=0} = \left(\frac{d}{dt}|<\psi(r, t)|\phi_0(r)>|^2\right)_{t=0} \tag{2.17}$$

This quantity is closely related to the expectation value of the Hamiltonian in the initial state, since we have

$$\begin{aligned} |<\phi_0|H|\phi_0>|^2 &= \left|\int \psi^*(r, 0)i\left(\frac{\partial\psi}{\partial t}\right)_{t=0} d^3r\right|^2 \\ &= \left(\frac{1}{2}\frac{d}{dt}|C_0(t)|^2\right)_{t=0} = \frac{1}{4}\Gamma^2 \end{aligned} \tag{2.18}$$

The second important quantity is the magnitude of the rate of change of the initial state wave function which is proportional to the expectation value of the square of the Hamiltonian in the initial state;

$$<\phi_0|H^2|\phi_0> = \int \psi^*(r, 0)H^2\psi(r, 0)\, d^3r = \left|\frac{dC_0}{dt}\right|^2_{t=0} \tag{2.19}$$

From equations (2.18) and (2.19) it follows that

$$<\phi_0|H^2|\phi_0> = \frac{1}{4}\Gamma^2 + \left[Im\left(\frac{dC_0}{dt}\right)_{t=0}\right]^2 \tag{2.20}$$

For a decaying system $<\phi_0|H^2|\phi_0>$ represents the rate of energy dissipation at $t = 0$, while Γ is the mean energy of the initial state. Now, the Hamiltonian of the system can be represented either as an operator in the coordinate (or momentum) space, or in terms of its matrix elements H_{nm} (2.13). In the latter form it is more convenient to work directly with the matrix equation (2.12), which for our solvable models reduces to differential-difference equations for $C_m(t)$ with the initial conditions (2.16). The temporal development of the initial state C_0, gives us the wave function at any other time t. In the following sections we will consider solvable models with four different types of Hamiltonians, which in the matrix representation can be written in the following forms:

$$\text{(a) } H_{nm} = \epsilon(m)\,\delta_{nm} + g \text{ or (b) } H_{nm} = \epsilon(m)\,\delta_{nm} + g(n + \tfrac{1}{2}) \tag{2.21}$$

$$H_{nm} = \epsilon(m)\delta_{nm} + ig[n\delta_{m+1,n} - m\delta_{m-1,n}] \tag{2.22}$$

$$H_{nm} = \epsilon(m)\delta_{nm} + ig[\delta_{m+1,n} - \delta_{m-1,n}] \tag{2.23}$$

and

$$H_{nm} = \epsilon(m)\delta_{nm} + g(\delta_{0n}\rho_m + \rho_n\delta_{0m}) \tag{2.24}$$

In these relations g is the coupling constant and ρ_m is a given vector. The matrix H_{nm} given by (2.21b) is not symmetric, however it corresponds to a Hermitian Hamiltonian as will be shown in the following section. Alternatively one can change the state vector and write (2.21b) as a symmetric Hamiltonian

$$H_{nm} = \epsilon(m)\delta_{nm} + g(n + \tfrac{1}{2})^{\frac{1}{2}}(m + \tfrac{1}{2})^{\frac{1}{2}} \tag{2.25}$$

3. *Models with Divergent H^2*

We want to study two models in this section, both having divergent matrix elements for H^2. But whereas in the first model the eigenvalues of the stationary states are well defined, in the second model one needs to renormalize the coupling constant in order to get finite results.

Let us consider the one-dimensional motion of a particle of mass $m = \frac{1}{2}$, with the Hamiltonian

$$H = \frac{\partial^2}{\partial x^2} + g\delta(x) \tag{3.1}$$

and subject to the periodic boundary condition

$$\psi(x = L) = \psi(x = -L) \tag{3.2}$$

If the initial state of the particle is given by an eigenfunction of $(-\partial^2/\partial x^2)$, e.g., if

$$\phi_0(x) = 1 \tag{3.3}$$

then the set of functions

$$\phi_m(x) = (2L)^{-\frac{1}{2}} \exp\left(\frac{i\pi m x}{L}\right) \tag{3.4}$$

can be used to expand the wave function $\psi(x, t)$

$$\psi(x, t) = \sum_{m=-\infty}^{+\infty} C_m(t) \exp\left(\frac{i\pi m}{L} x\right) \tag{3.5}$$

The Hamiltonian (3.1) and the wave function (3.5) yield the following differential equation for $C_n(t)$

$$i\frac{dC_n}{dt} = \frac{\pi^2 n^2}{L^2} C_n(t) + \frac{g}{L} \sum_{m=-\infty}^{+\infty} C_m(t) \tag{3.6}$$

with the initial conditions that at $t = 0$,

$$C_0 = 1 \quad \text{and} \quad C_n = 0 \quad n \neq 0 \tag{3.7}$$

Using the method of the previous section and writing

$$C_n(t) = \sum_E C_n(E) C_0^*(E) e^{-iEt} \tag{3.8}$$

we obtain the eigenvalue problem

$$C_n(E) = \frac{g}{L(E - \pi^2 n^2/L^2)} \sum_{m=-\infty}^{+\infty} C_m(E) \tag{3.9}$$

Summing both sides of this equation over n, we find a transcendental equation for the eigenvalues which is very similar to the eigenvalue equation for the Wigner-Weisskopf model

$$g^{-1} = E^{-\frac{1}{2}} \cot(E^{\frac{1}{2}} L) \tag{3.10}$$

In order to construct the normalized eigenvectors, we divide $C_n(E)$ (3.9) by $C_0(E)$, with the result that all C_n's can be related to C_0;

$$C_n(E) = E C_0(E) \left(E - \frac{\pi^2 n^2}{L^2}\right)^{-1} \tag{3.11}$$

Now from equation (3.8) and the normalization condition, viz.,

$$\int_{n=-\infty}^{+\infty} C_n(E) C_n^*(E') = \delta_{EE'} \tag{3.12}$$

it follows that

$$\sum_{n=-\infty}^{+\infty} |C_n(t)|^2 = \sum_E |C_0(E)|^2 = 1 \tag{3.13}$$

Thus, to satisfy the initial conditions (3.7), we need to normalize $C_n(E)$

according to (3.12) or use equation (3.13). We determine $C_0(E)$ by substituting (3.11) in (3.13)

$$|C_0(E)|^2 = (d\Delta/dE)^{-1} \tag{3.14}$$

where

$$\Delta(E) = LE^2\,[g^{-1} - E^{\frac{1}{2}}\cot(E^{\frac{1}{2}}L)] \tag{3.15}$$

With the help of the function $\Delta(E)$ we can write $C_0(t)$, equation (3.8) in terms of a contour integral (Haake and Weidlick, 1968; Razavy and Henley, 1970)

$$C_0(t) = \frac{1}{2\pi i}\oint [e^{-izt}/\Delta(z)]\,\mathrm{d}z \tag{3.16}$$

This integral is similar to the time evolution equation in the Wigner-Weisskopf model, and it can be evaluated approximately. The result indicates that the initial state C_0 decays exponentially into the other states $C_n(t)$ (see, for instance, Haake and Weidlich, 1968).

The model above corresponds to the Hamiltonian (2.21a). The second model with a Hamiltonian similar to (2.21b) is more interesting, since, if we apply the perturbation theory to calculate transition probabilities, we find that the first order term is well defined and finite, but the second and higher order terms are divergent. Thus the original Hamiltonian must be renormalized to insure the finiteness of the results to all orders. This model is a one-dimensional quantum mechanical system. Let us consider a particle of mass $\frac{1}{2}$ which is constrained to move on a cylinder of radius L. The Hamiltonian describing the motion of the particle is given by

$$H = -\frac{1}{L^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{g_0}{L}\,\delta(\cos\theta - 1) \tag{3.17}$$

where g_0 is the strength of the potential. The resulting wave function of the time dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{3.18}$$

can be expanded in terms of the Legendre polynomials

$$\psi(\theta, t) = \sum_{l=0}^{\infty} C_l(t)P_l(\cos\theta) \tag{3.19}$$

By substituting (3.19) in (3.18) and using the orthogonality of the Legendre polynomials we find a differential equation for $C_m(t)$

$$i\frac{dC_m}{dt} = \frac{m(m+1)}{L^2}C_m + \frac{g_0}{L}(m+\tfrac{1}{2})\sum_{l=0}^{\infty} C_l \tag{3.20}$$

As before we solve this equation by Fourier transform method, equation (3.8), and we find that the eigenvectors satisfy the algebraic equation

$$\left(E - \frac{m(m+1)}{L^2}\right) C_m = \frac{g_0}{L}(m + \tfrac{1}{2}) \sum_{l=0}^{\Lambda} C_l \tag{3.21}$$

In the last term of this equation we have introduced a cut-off Λ (which is a large positive integer) to make the results finite. The equation for the characteristic values can be obtained in the same way as described earlier and it contains the cut-off Λ;

$$\sum_{m=0}^{\Lambda} \frac{g_0 L(m + \frac{1}{2})}{EL^2 - m(m+1)} = 1 \tag{3.22}$$

If the particular eigenvalue of equation (3.22) which we want to determine is close to $l(l + 1)/L^2$, where l is an integer, then we write

$$\frac{1}{g_0 L} = \frac{l + \frac{1}{2}}{EL^2 - l(l+1)} - \sum_{m=1}^{\Lambda}{}' \frac{1}{m} + \frac{1}{2L^2 E} + \sum_{m=1}^{\Lambda}{}' \frac{EL^2 - \frac{1}{2}m}{m[EL^2 - m(m+1)]} \tag{3.23}$$

where the prime on Σ indicates that the term $m = l$ is excluded from the sum. As Λ becomes very large, $\Sigma' 1/m$ approaches $\log \Lambda$, and thus we get

$$\frac{1}{g_0 L} = \frac{l + \frac{1}{2}}{EL^2 - l(l+1)} - \log \Lambda - \text{finite terms} \tag{3.24}$$

From this relation we find E

$$E = \frac{l(l+1)}{L^2} + \frac{(l + \frac{1}{2})(g_0/L)}{1 + g_0 L(\log \Lambda + \text{finite terms})} \tag{3.25}$$

Now if we assume that $g_0 L$ is very small, we can determine E by expanding the right hand side of (3.25) in powers of $g_0 L$. Thus

$$E = \frac{l(l+1)}{L^2} + \frac{g_0}{L}(l + \tfrac{1}{2}) - g_0^2 (l + \tfrac{1}{2})[\log \Lambda + \text{finite terms}] \tag{3.26}$$

Equations (3.25) and (3.26) show that the radius of convergence of the perturbation series is $(\log \Lambda)^{-1}$ and tends to zero as Λ becomes very large. The normalized eigenvectors can be found from equation (3.21), and in particular $|C_0(E)|^2$ is given by

$$|C_0|^2 = \left[\sum_{m=0}^{\infty} \frac{(2m+1)^2 E^2 L^4}{[EL^2 - m(m+1)]^2}\right]^{-1} \tag{3.27}$$

If initially, all C_n's are zero except C_0 which is equal to one, then the time dependence of the state $C_0(t)$ is determined by equation (3.8). Thus we can write

$$C_0(t) = \sum_E \frac{e^{-iEt}}{D(E)} \tag{3.28}$$

where

$$\begin{aligned} D(E) &= E^2L^4 \sum_{m=0}^{\infty} \frac{(2m+1)^2}{(EL^2 - m^2 - m)^2} \\ &= E^2L^2 \left\{ 2\pi(1+4EL^2)^{-\frac{1}{2}} \tan\left[\frac{\pi}{2}(1+4EL^2)^{\frac{1}{2}}\right] \right. \\ &\quad \left. + \pi^2/\cos^2\left[\frac{\pi}{2}(1+4EL^2)^{\frac{1}{2}}\right]\right\} \end{aligned} \tag{3.29}$$

We note that $(1 + 4EL^2)^{1/2}$ cannot be equal to an odd integer, a condition that follows from the solution of the eigenvalue equation (3.22). Also from the last equation it is evident that a cut-off is not necessary for calculating $C_0(E)$, however, equation (3.28) shows that $C_0(t)$ will not be a well defined function unless the cut-off Λ is introduced. We can renormalize the coupling constant g in the following way (Peres, 1969). Let us define g by the relation

$$\frac{1}{gL} = \frac{1}{g_0L} + \sum_{m=1}^{\Lambda} \frac{1}{m} = \frac{1}{g_0L} + S(\Lambda) \tag{3.30}$$

and then solve for g_0

$$g_0 = g/1 - g[S(\Lambda)] \tag{3.31}$$

The renormalized Hamiltonian can be written as

$$H = -\frac{1}{L^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{g}{1 - gLS(\Lambda)}\delta(\cos\theta - 1) \tag{3.32}$$

It can be shown that the perturbation given by the last term in equation (3.32) yields finite results to all orders in g (Peres, 1969).

4. *Motion of a Particle in a Periodic Potential*

If the initial state of a system is an eigenstate of the unperturbed Hamiltonian, and then the system is perturbed by a periodic potential, this state, under certain conditions, decays as Bessel function of time. Thus the temporal development of the state is very similar to the decay of the motion of a displaced particle in the problem of loaded string of infinite length in classical dynamics (Schrödinger, 1914). Consider the motion of a particle of mass $m = \frac{1}{2}$ moving under the action of a periodic force $2g\cos(\pi x/L)$ in a straight line, and subject to the boundary conditions

$$\psi(x = L) = \psi(x = -L) = 0 \tag{4.1}$$

The time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + 2g\cos\left(\frac{\pi x}{L}\right)\psi \tag{4.2}$$

can be transformed to a differential-difference equation with the expansion of ψ in terms of the unperturbed set of wave functions

$$\phi_n(x) = L^{-\frac{1}{2}}\sin\left(\frac{n\pi x}{L}\right) \tag{4.3}$$

Thus if $C_n(t)$'s are the coefficients of the expansion, we have

$$i\frac{dC_n}{dt} = \frac{n^2\pi^2}{L^2}C_n + g(C_{n+1} + C_{n-1}) \quad n = 1, 2, \ldots \tag{4.4}$$

The initial conditions on the solutions of this set are related to the initial state. Assuming that at $t = 0$,

$$\psi(x, 0) = L^{-\frac{1}{2}}\sin\left(\frac{\pi x}{L}\right) \tag{4.5}$$

then we have

$$C_1(0) = 1, \qquad C_n(0) = 0, \qquad n \neq 1 \tag{4.6}$$

The stationary solutions of (4.2) are given by the solution of the Mathieu equation

$$\frac{d^2\psi}{dx^2} + \left[k^2 - 2g\cos\left(\frac{\pi x}{L}\right)\right]\psi = 0 \tag{4.7}$$

with the boundary condition (4.1). From the set of four Mathieu functions, we choose the one that is odd about the origin and is even about the point $x = \frac{1}{2}L$, since these are the symmetries of the initial wave function. With these symmetries, we find $\psi(x, k)$ to be expressed by the Mathieu function (Morse and Feshbach, 1953)

$$\psi_m(x, k_m) = N_m SO_{2m}\left[g, \cos\left(\frac{\pi x}{2L}\right)\right] = N_m \sum_{n=1}^{\infty} B_{2n}^0(g, 2m)\sin\left(\frac{n\pi x}{L}\right) \tag{4.8}$$

where N_m is the normalization constant

$$N_m = \left\{L\sum_{n=0}^{\infty}[B_{2n}^0(g, 2m)]^2\right\}^{-\frac{1}{2}} \tag{4.9}$$

and where

$$\sum_n 2nB_{2n}^{(0)} = 1 \tag{4.10}$$

This solution, as g tends to zero, approaches the limit

$$SO_{2m} \xrightarrow[g\to 0]{} \sin\left(\frac{m\pi x}{L}\right) \tag{4.11}$$

and the corresponding eigenvalue becomes $k_m = m\pi/L$. We expand the initial wave function $\psi(x, 0)$ in terms of $\psi_m(x, k_m)$

$$L^{-1/2} \sin\left(\frac{\pi x}{L}\right) = N_m \sum_m C_m SO_{2m}\left[g, \cos\left(\frac{\pi x}{2L}\right)\right] \tag{4.12}$$

The coefficient of expansion C_m is determined from (4.12), using the orthogonality of Mathieu functions

$$C_m = B_2^0(g, 2m)\left\{L \sum_{n=0}^{\infty} [B_{2n}^0(g, m)]^2\right\}^{-\frac{1}{2}} \qquad m = 1, 2, \ldots \tag{4.13}$$

Having found the coefficients as functions of the eigenvalues, we can obtain the time evolution of the original state C_1 by substituting (4.13) in equation (3.8)

$$C_1(t) = \sum_{k_b} e^{-ik_b^2 t} |C_1(k_b)|^2 \tag{4.14}$$

where k_b^2 are the eigenvalues of the Mathieu equation associated with the eigenfunctions SO_{2m}. When L becomes very large (4.4) reduces to the following equation

$$i\frac{dC_n}{dt} = g(C_{n+1} - C_{n-1}) \tag{4.15}$$

Changing the function C_m to

$$C_m(t) = i^{m-1} J_{m-1}(-2gt) \tag{4.16}$$

we observe that

$$\frac{dJ_{m-1}}{dt} = -g(J_{m-1} - J_{m+1}) \qquad m = 1, 2, \ldots \tag{4.17}$$

Therefore in this limit the solution of (4.4) reduces to Bessel function and $C_1(t)$ decays as

$$C_1(t) = J_0(-2gt) \tag{4.18}$$

A simpler but an unrealistic model is the one in which the matrix elements of the Hamiltonian are given by

$$H_{nm} = \frac{n\pi}{\mu L^2}\delta_{nm} - ig[\delta_{n+1,m} - \delta_{n-1,m}] \tag{4.19}$$

This Hamiltonian corresponds to the first order wave equation for a particle of mass $\frac{1}{2}\mu$, i.e.

$$\frac{\partial\psi}{\partial t} = -\frac{1}{\mu L}\frac{\partial\psi}{\partial x} + 2ig \sin\left(\frac{\pi x}{L}\right)\psi \tag{4.20}$$

with the boundary condition

$$\psi(L) = \psi(-L) \tag{4.21}$$

Here the expansion coefficients of $\psi(x, t)$ satisfy the Schrödinger equation

$$i\frac{dC_n}{dt} = \sum_m H_{nm} C_m \tag{4.22}$$

From the solution of (4.20), we find the result of integrating (4.22), viz.

$$C_n(t) = \frac{1}{2L}\int_{-L}^{+L} \exp\left\{\frac{4ig\mu L^2}{\pi}\sin\left(\frac{\pi t}{2\mu L^2}\right)\sin\frac{\pi}{L}\left(x - \frac{t}{2\mu L}\right) - \frac{in\pi x}{L}\right\} dx \tag{4.23}$$

For large values of L the integral in (4.23) can be evaluated in closed form

$$\lim_{L\to\infty} C_n(t) \to J_n(-2gt) \tag{4.24}$$

So in this limit the initial state C_0 decays like a Bessel function.

5. *Interacting Systems*

The following model is very similar to the models of Wigner–Weisskopf and of Martin. In this model there are two interacting systems. The state of the first system is described by the wave function $C_0(t)$, which depends only on time, and the second system is a particle of mass $\frac{1}{2}$ and the wave function $\psi(r, t)$. The equations of motion of the two systems are:

$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi = g\rho(r)C_0(t) \tag{5.1}$$

and

$$\left[i\frac{d}{dt} - \mu\right]C_0(t) = g\int\rho(r)\psi(r, t)\,d^3r \tag{5.2}$$

where g and μ are real constants. We assume that $\rho(r)$ is non zero only within a very short range r_0, and that it is normalized

$$\int\rho(r)\,d^3r = 1 \tag{5.3}$$

To solve the coupled set of equations we find the Fourier transform of equations (5.1) and (5.2)

$$(k^2 + \nabla^2)\psi(r, k) = g\rho(r)C_0(k) \tag{5.4}$$

$$(k^2 - \mu^2)C_0(k) = g\int\rho(r)\psi(r, k)\,d^3r \tag{5.5}$$

Taking $\psi(r, k)$ to be

$$\psi(r, k) = \frac{1}{r}\sin\left(kr + \eta(k)\right) \tag{5.6}$$

where $\eta(k)$ is the phase shift, then after substitution and some reduction we get

$$C_0(k) = \frac{g}{k^2 - \mu}\left(k \cos \eta + \frac{1}{r_0} \sin \eta\right) \tag{5.7}$$

and

$$k \cot \eta(k) = -r_0^{-1} - \frac{4\pi}{g^2}(k^2 - \mu) \tag{5.8}$$

Now let us assume that at $t = 0$, the initial conditions are

$$C_0(0) = 1, \qquad \psi(r, 0) = 0 \tag{5.9}$$

Therefore the quantity $|\langle C_0(0)|C_0(t)\rangle|^2$ represents the probability that at the time $t(t > 0)$ the state has not decayed. This probability can directly be related to the interaction parameters, since from (5.7) and (5.8) it follows that

$$\langle C_0(0)|C_0(t)\rangle = \int_0^\infty \frac{k^2 \exp(-ik^2 t)\, dk}{\epsilon(k^2)} \tag{5.10}$$

where

$$\epsilon(k^2) = \frac{g^2 k^2}{16\pi^2} + \left[\frac{g}{4\pi r_0} + (k^2 - \mu)\right]^2 \tag{5.11}$$

In the initial state the mean energy of the state C_0 is given by

$$\langle C_0(0)|k^2|C_0(0)\rangle = \int_0^\infty \frac{k^4\, dk}{\epsilon(k^2)} \tag{5.12}$$

Since the integral in (5.12) is divergent, the initial state is not realizable. Furthermore, $|\langle C_0(0)|C_0(t)\rangle|^2$ has infinite derivative at $t = 0$, and the decay curve exhibits a cusp. If we enclose this system in a large sphere of radius L, and assume that the wave function $\psi(r, t)$ remains finite inside the volume and vanishes on the surface, then we can expand $\psi(r, t)$ and $\rho(r)$ in terms of the set of functions $\sin(m\pi x/L)$

$$\psi(r, t) = \frac{1}{\sqrt{2\pi L}} \sum_{m=-\infty}^{+\infty} \frac{1}{r} C_m \sin\left(\frac{m\pi r}{L}\right) \tag{5.13}$$

and

$$\rho(r) = \frac{1}{\sqrt{2\pi L}} \sum_{m=-\infty}^{+\infty} \frac{1}{r} \rho_m \sin\left(\frac{m\pi r}{L}\right) \tag{5.14}$$

The Hamiltonian of the coupled system is given by

$$H_{nm} = \frac{m^2\pi^2}{L^2}\delta_{nm} + \mu\delta_{0n}\delta_{0m} + g(\rho_m\delta_{0n} + \rho_n\delta_{0n}) \tag{5.15}$$

and the time dependent Schrödinger equation

$$i\frac{dC_n}{dt}=\sum_{m=-\infty}^{+\infty} H_{nm}C_m \tag{5.16}$$

corresponds to equations (5.1) and (5.2). A somewhat similar model, but with the delayed action, can be constructed in which both $\langle C_0(0)|H|C_0(0)\rangle$ and $\langle C_0(0)|H^2|C_0(0)\rangle$ are finite (Razavy, 1967).

Now let us consider the time-dependent many-body problem. We study two simple models to gain some insight into the problem of the decay times and periods and their relation to the properties of the forces and the number of particles. The Bassichis–Foldy model is a one dimensional many-boson system with the Hamiltonian

$$\begin{aligned}H=a_1^\dagger a_1+a_3^\dagger a_3+g[a_2^\dagger a_2(a_1^\dagger a_1+a_3^\dagger a_3)+a_2^2a_1^\dagger a_3+a_2^{\dagger 2}a_1a_3]\\ -Fga_1^\dagger a_1a_3^\dagger a_3\end{aligned} \tag{5.17}$$

where $a_1^\dagger$, $a_2^\dagger$, and $a_3^\dagger$ are creation operators for the states of positive, zero and negative momenta respectively (Bassichis and Foldy, 1964). The dimensionless constants g and Fg are the strengths of interactions. This Hamiltonian commutes with the following two operators:

(a) The number operator N

$$N=a_1^\dagger a_1+a_2^\dagger a_2+a_3^\dagger a_3 \tag{5.18}$$

and

(b) The difference between the number of particles in the states 1 and 3, i.e.,

$$\Delta=a_1^\dagger a_1-a_3^\dagger a_3 \tag{5.19}$$

The many-body wave function which describes the evolution of the system in time is the solution of the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t}=H\Psi \tag{5.20}$$

This wave function depends on the number of particles in the three different states, or because of the constants of motion N and Δ, on the number n of particles in the state 3. We have different sectors for this model, each sector is specified by N and Δ and an independent wave function $\Psi(N,\Delta,n)$, where n is the eigenvalue of the operator $a_3^\dagger a_3$. The time dependent Schrödinger equation can be transformed to a differential-difference equation by substituting H from (5.17) in (5.20) to get

$$\begin{aligned}&g[(N-\Delta-2n+2)(N-\Delta-2n+1)(\Delta+n)n]^{\frac{1}{2}}C_{n-1}\\ &\quad+\left[\Delta+2n+g(N-\Delta-2n)(\Delta+2n)-Fgn(n+\Delta)-i\frac{d}{dt}\right]C_n\\ &\quad+g[(N-\Delta-2n-1)(N-\Delta-2n)(\Delta+n+1)(n+1)]^{\frac{1}{2}}C_{n+1}=0\end{aligned} \tag{5.21}$$

This set uncouples at $n = \frac{1}{2}(N - \Delta)$, and therefore we set

$$C_{-1} = C_{\frac{1}{2}(N-\Delta)+1} = 0 \tag{5.22}$$

For the initial conditions we assume that

$$C_0(t = 0) = 1, \qquad C_n(t = 0) = 0, \qquad n \neq 0 \tag{5.23}$$

In the sector where

$$N = \Delta + 2 \tag{5.24}$$

the time dependent Schrödinger equation simplifies to

$$\left[N - 2 + 2g(N - 2) - i\frac{d}{dt}\right] C_0 = -g[2(N - 1)]^{\frac{1}{2}} C_1 \tag{5.25}$$

and

$$\left[N - Fg(N - 1) - i\frac{d}{dt}\right] C_1 = -g[2(N - 1)]^{\frac{1}{2}} C_0 \tag{5.26}$$

Thus in this sector the quantity $|C_0|^2$ is the probability of finding $N - 2$ particles in the state 1, and two particles in the state 2 while C_1 represents the amplitude for having $N - 1$ particles in the state 1, and one particle in 3. By substituting $C_n(t) = C_n(\omega)e^{i\omega t}$ we find the stationary solutions of equations (5.25) and (5.26). The eigenvalues of the problem are given by the roots of the quadratic equation in ω

$$[(N - 2)(1 + 2g) - \omega]\,[N - Fg(N - 1) - \omega] = 2g^2(N - 1) \tag{5.27}$$

If we denote the eigenvectors by $C_0(\omega)$ and $C_1(\omega)$ we have the normalization condition

$$C_0^2(\omega) + C_1^2(\omega) = 1 \tag{5.28}$$

We are interested in the behaviour of the system for large N, but to find the asymptotic form of the eigenvectors we need to exclude the point where F becomes equal to F_T

$$F_T = -\frac{2}{g}(gN - 2g - 1)/(N - 1) \tag{5.29}$$

This point corresponds to the value of F, which makes both of the states of the system equally probable, i.e.

$$C_0^2(\omega) = C_1^2(\omega) = \tfrac{1}{2} \tag{5.30}$$

For all other values of F, and for $N \gg 1$, we have two eigenvalues

$$\omega_{\pm} \simeq N\left[\left(1 + g - \frac{1}{2}Fg\right) \pm g\left\{\left(1 + \frac{1}{2}F\right)^2 + \frac{2}{N}\right\}^{\frac{1}{2}}\right] \tag{5.31}$$

and the eigenvectors

$$C_0^2(\omega_-) = C_1^2(\omega_+) = 1 - [2N(1 + \tfrac{1}{2}F)^2]^{-1} + \ldots \tag{5.32}$$

and

$$C_1^2(\omega_-) = C_0^2(\omega_+) = [2N(1 + \tfrac{1}{2}F)^2]^{-1} + \ldots \tag{5.33}$$

In the last two relations terms proportional to N^{-2} and higher powers of N^{-1} have been neglected. From the eigenvalue equation (5.27) it follows that when $g > 0$, and $F > F_T$, ω_- is less than ω_+, and thus ω_- is the ground state energy of the system. Using equations (5.33) and (5.34), we find that $C_0^2(\omega_-) > C_1^2(\omega_-)$, and therefore the ground state of the system is the state with two particles in the state 2 and $N - 2$ particles in 1. By taking $F < F_T$ we find that $\omega_+ < \omega_-$, and hence ω_+ is the ground state energy and $C_1^2(\omega_+)$ represent the probability of finding the system in its lowest energy state. The time dependent states of the system are given by

$$C_j(t) = C_0^*(\omega_+)C_j(\omega_+)e^{-i\omega_+t} + C_0^*(\omega_-)C_j(\omega_-)e^{-i\omega_-t}, \qquad j = 0, 1 \tag{5.34}$$

A more interesting case of the model given by (5.17) is the limit when N becomes very large and g very small, in such a way that gN remains finite. If we look at the sector with $\Delta = 0$, in this limit, we find that the differential-difference equation (5.1) reduces to

$$i\frac{dC_n}{dt} = 2n(1 + gN)C_n + gN\{nC_{n-1} + (n + 1)C_{n+1}\} \tag{5.35}$$

where the index n now can be any nonnegative integer. For the initial conditions we assume that at $t = 0$, only C_0 is different from zero, i.e.

$$C_0(t = 0) = 1, \qquad C_n(t = 0) = 0, \qquad n \neq 0 \tag{5.36}$$

To solve this problem let us first consider the solution of the time-dependent Schrödinger equation with the fictitious Hamiltonian

$$H = -\frac{i}{\mu L}\frac{\partial}{\partial x} + \frac{gL}{\pi}\left(e^{-i(\pi x/L)}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}e^{i(\pi x/L)}\right) \tag{5.37}$$

Let us also assume that the boundary and the initial conditions are given by

$$\psi(L) = \psi(-L) \tag{5.38}$$

and

$$\psi(x, t = 0) = (2L)^{-\frac{1}{2}} \tag{5.39}$$

The wave equation obtained from the Hamiltonian (5.37) is a first order partial differential equation

$$\frac{\partial\psi}{\partial t} + \left[\frac{1}{\mu L} + \frac{2L}{\pi}g\sin\left(\frac{\pi x}{L}\right)\right]\frac{\partial\psi}{\partial x} + ge^{i(\pi x/L)}\psi = 0 \tag{5.40}$$

The solution of this equation satisfying the initial condition (5.39) and the boundary condition (5.38) is

$$\psi(x, t) = (2L)^{-\frac{1}{2}} \exp\left(\frac{i\pi}{2\mu L^2} t\right) \left\{\cosh \gamma t + \frac{g}{\gamma}\left(e^{i(\pi x/L)} + \frac{i\pi}{2\mu L^2 g}\right) \times \sinh \gamma t\right\}^{-1} \tag{5.41}$$

where

$$\gamma^2 = g^2 - \frac{\pi^2}{(2\mu L^2)^2} \tag{5.42}$$

Now if we expand $\psi(x, t)$ in terms of the orthogonal set $(2L)^{-1/2} \exp(in\pi x/L)$, i.e.

$$\psi(x, t) = \sum_{n=0}^{\infty} D_n(t)(2L)^{-\frac{1}{2}} \exp\left(\frac{in\pi x}{L}\right) \tag{5.43}$$

we find that $D_n(t)$ satisfies the set of equations

$$\frac{dD_n}{dt} = -\frac{in\pi}{\mu L^2} D_n + g[(n+1)D_{n+1} - nD_{n-1}] \tag{5.44}$$

Note that in (5.43) n is restricted to nonnegative integers. This guarantees that the kinetic energy in this model is a positive-definite quantity. To solve equation (5.44) for $D_n(t)$, we expand $\psi(x, t)$ (5.41) in powers of $\exp(i\pi x/L)$, and compare the result with (5.43). Thus we find

$$D_n(t) = \left(-\frac{g}{\gamma}\right)^n \exp\left(\frac{i\pi t}{2\mu L^2}\right)(\sinh \gamma t)^n \left(\cosh \gamma t + \frac{i\pi}{2\pi L^2 \gamma} \sinh \gamma t\right)^{-n-1} \tag{5.45}$$

When the constants μ, L and g are such that the quantity

$$\omega^2 = \frac{\pi^2}{(2\mu L^2)^2} - g^2 \tag{5.46}$$

is positive, then $D_n(t)$ is a periodic function of time, i.e.

$$D_n(t) = \left(-\frac{g}{\omega}\right)^n \exp\left(\frac{i\pi t}{2\mu L^2}\right) \sin^n \omega t \left(\cos \omega t + \frac{\pi i}{2\mu L^2} \sin \omega t\right)^{-n-1} \tag{5.47}$$

For the critical case of $g^2 = \pi^2/(2\mu L^2)^2$, we have nonexponential damping:

$$D_n(t) = e^{igt}(-gt)^n(1 + igt)^{-n-1} \tag{5.48}$$

Now we return to our original set of equations for C_n, and we compare equations (5.35) and (5.44), to find $C_n(t)$

$$C_n(t) = i^n D_n(t) = \left(-\frac{g}{\nu}\right)^n \exp[i(1+Ng)t]\ \sin^n \nu t \times \\ \times \left(\cos \nu t + \frac{i(1+Ng)}{\nu} \sin \nu t\right)^{-n-1} \tag{5.49}$$

where

$$\nu = (1 + 2Ng)^{\frac{1}{2}} \tag{5.50}$$

Thus depending on the sign of g, C_n can be a trigonometric or a hyperbolic function of time. For the following many particle Hamiltonian

$$H = ig(a_2^{\dagger 2} a_3 a_1 - a_2{}^2 a_3^\dagger a_1^\dagger) \tag{5.51}$$

which has the same constants of motion N and Δ as the preceding problem (5.18) and (5.19), the time dependent Schrödinger equation in the limit of very large N and for $\Delta = 0$, reduces to

$$\frac{dC_n}{dt} = Ng[(n+1)C_{n+1} - nC_{n-1}] \tag{5.52}$$

This is a special case of equation (5.44). Thus for this model we have

$$|C_0(t)|^2 = (\cosh Ngt)^{-2} \tag{5.53}$$

6. *Penetration Through Phase-Equivalent Potential Barriers*

A problem closely related to the decay of an initial state is that of the leakage of a wave packet through a potential barrier. Let us assume that the initial wave packet is $\phi_0(r)$ and the potential barrier is represented by $v(r)$, then $\psi(r, t)$ (2.8) gives us the amplitude of the wave at the time t and about the point r. The stationary state solution $\psi(E, r)$ is the characteristic function of the Hamiltonian

$$H = -\nabla^2 + v(r) \tag{6.1}$$

with the eigenvalue E. Now suppose that $v(r)$ is replaced by $\tilde{v}(r, r')$ where $v(r)$ and $\tilde{v}(r, r')$ are phase equivalent, i.e., they give rise to identical set of phase shifts for all energies and all partial waves. We may inquire whether the decay law is different for the two potentials, if we assume the same initial wave packet for the time dependent Schrödinger equation. In other words we want to compare the problem of leakage of a given initial state $\phi_0(r)$ through two different, but phase-equivalent potential barriers. It is intuitively clear that for small distances, i.e., points just outside the barrier, the time dependent wave function $\psi(r, t)$ will be different from $\psi(r, t)$, the latter being the wave function for the Hamiltonian with the potential $\tilde{v}(r, r')$. The result, for large distances from the barrier, however, is nontrivial.

There are different ways of constructing phase equivalent potentials, but we choose the simplest method, that proposed by Coester et al. (Coester, 1970), to study the decay law. From a potential $\tilde{v}(r)$, we can construct another phase equivalent nonlocal potential $\tilde{v}(r, r')$, using the following unitary transformation

$$U(r, r') = \delta(r - r') - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l(r) g_l(r') \hat{Y}_{lm}(\hat{r}) \hat{Y}_{lm}(\hat{r}') \tag{6.2}$$

where $\hat{r}$ is the unit vector in the direction of r, and $g_l(r)$ is a short range function of r. The requirement that $U_0(r, r')$ is a unitary operator imposes the following condition on $g_l(r)$

$$\int_0^{\infty} g_1{}^2(r) r^2 dr = 2 \tag{6.3}$$

The potential $\tilde{v}(r, r')$ turns out to be a nonlocal operator (Coester, 1970),

$$\tilde{v} = U(-\nabla^2 + v)U^{\dagger} + \nabla^2 \tag{6.4}$$

and the corresponding wave function is related to $\psi(r, E)$ by the integral

$$\tilde{\psi}(r, E) = \int U(r, r') \psi(r', E) d^3 r' \tag{6.5}$$

For the simple case where $U(r, r')$ is given by

$$U(r, r') = \delta(r - r') - \frac{1}{4\pi} g(r) g(r') \tag{6.6}$$

the only partial wave affected by the transformation is the S-wave. Denoting the original wave function for zero angular momentum state by $u(r)$ and the transformed wave function by $\tilde{u}(r)$, we find from equation (6.5) that

$$\tilde{u}(r, E) = u(r, E) - g(r) \int_0^{\infty} g(r_1) u(r_1, E)\, dr_1 \tag{6.7}$$

If we write the time dependent solution (2.8) first for $\tilde{u}(r, t)$ and then for $\tilde{u}(r, t)$ with the same initial state $\phi_0(r)$, then by subtracting $\tilde{u}(r, t)$ from $u(r, t)$, and by substituting from (6.7), after reduction we get

$$u(r, t) - \tilde{u}(r, t) = \int_0^{\infty} \phi_0(r')\, dr' \left\{ \int_0^{\infty} g(r_2) [g(r') \Delta(r, r_2, t) + g(r) \Delta(r_2, r', t)]\, dr_2 - g(r) g(r') \int_0^{\infty} g(r_1) g(r_2) \Delta(r_1, r_2, t)\, dr_1 dr_2 \right\} \tag{6.8}$$

where

$$\Delta(r, r_2, t) = \int_0^{\infty} e^{-iEt} u(r, E) u^*(r_2, E)\, dE \tag{6.9}$$

Now let us consider those transformations for which $g(r)$ decreases exponentially or faster as r tends to infinity, then for large r, we can ignore the second and third terms on the right hand side of equation (6.8). Thus for large r, we have

$$u(r, t) - \tilde{u}(r, t) \xrightarrow[r \to \infty]{} \int_0^{\infty} \Delta(r, r_2, t) g(r_2)\, dr_2 \int_0^{\infty} \phi_0(r') g(r')\, dr' \tag{6.10}$$

From this relation, it is easy to show that the current and the density for large distances from the barrier will be significantly different for the two potentials. For example, if we assume that $v(r) = 0$, i.e., in the absence of any barrier for $u(r, t)$, but with a $g(r)$ given by

$$g(r) = \left(\frac{32\mu^2}{\pi}\right)^{\frac{1}{4}} \exp(-\mu^2 r^2) \tag{6.11}$$

we find observable difference between the currents (or the densities) associated with $u(r, E)$ and $\tilde{u}(r, E)$. In this example $u(r, E)$ is the wave function of a free particle

$$u(r, E) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} E^{-\frac{1}{4}} \sin(E^{1/2} r) \tag{6.12}$$

and the propagator Δ can be found from (6.9),

$$\Delta(r, r_2, t) = (\pi i t)^{-\frac{1}{2}} \left\{\exp\left[\frac{i}{4t}(r - r_2)^2\right] - \exp\left[\frac{i}{4t}(r + r_2)^2\right]\right\} \tag{6.13}$$

By substituting (6.11) and (6.13) in (6.10), we obtain

$$\begin{aligned} u(r, t) - \tilde{u}(r, t) \underset{r\to\infty}{\longrightarrow} & \frac{-8(1-i)\mu}{2\pi t^{\frac{1}{2}}} \exp\left(\frac{ir^2}{4t}\right) \times \\ & \times \frac{r}{(4t\mu^2 - i)}\, {}_1F_1\left[1, \frac{3}{2}, \frac{-r^2}{4t(4t\mu^2 - i)}\right] \int_0^\infty e^{-\mu r_2^2} \phi_0(r_2)\, dr_2 \\ \approx & \frac{-16(1-i)}{2\pi} e^{(ir^2/4t)} \frac{t^{\frac{1}{2}}}{r} \int_0^\infty e^{-\mu^2 r_2^2} \phi_0(r_2)\, dr_2 \end{aligned} \tag{6.14}$$

This result may be interpreted in the following way: Since in the penetration through a barrier, as formulated in this section, there is considerable overlap between the initial wave function and the potential. Therefore, the flow of the probability current depends on the shape of the barrier. Thus, one may conclude that in general the relation between the scattering phase shifts and the lifetime and the mode of decay of an unstable state (or particle) is not unique, and, unless other constraints are added to the problem, one can have many possible solutions.

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